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对称扰动与纬向基流的相互作用*
II. 粘性波包的发展与弥散

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摘 要 该文对对称扰动与纬向基流的相互作用理论进行了深入的研究。这里是该文的第二部分，主要讨论粘性波包的发展与弥散。在对称扰动波包的动力学方程组中引进粘性作用($Pr=1$)可以发现，粘性耗散及扩散导致了"虚群速度(C_{gy}^{i}, C_{gz}^{i})"的产生，它对波包的发展有着比较重要的影响。此外，各种稳定度参数的时空变化对扰动波包的发展具有重要作用。在一定的条件下，还可以产生波包的不稳定现象。

关键词 波-流相互作用 粘性波包

1 引言

有关扰动波包的发展问题，我国学者曾庆存首先对正压大气、斜压大气中的Rossby波包的演变和发展作出了开拓性的研究[1~4]。他用波包表示天气尺度扰动，并用WKBJ近似，揭示出扰动发展与否的普遍判据，以及所有波的特征（如波振幅、波长、相速等）随时间变化的规律性，从而克服了传统方法（正交模方法，积分特征分析法等）在处理非均匀基本气流情况下扰动稳定性时所遇到的困难，得出了符合天气学事实的规律。对于非定常的弱热成风平衡的大尺度背景，也可以应用WKBJ方法分析二维动量无辐散近似下的对称扰动动力学方程组，结果发现中尺度扰动波包对称发展的原因是基本场的非定常性以及热成风偏差[5]，并进一步可以分析得出流场非均匀性、各种稳定度参数的时空变化对重力惯性波包发展的影响[6~7]。本文在前人工作的基础上，引进粘性耗散及扩散效应，考察其对波包发展的作用。

2 数学模型

在连续方程采用均匀不可压缩假设条件下，不考虑凝结潜热加热的粘性大气运动完全方程组为

$$\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv+F_x, \tag{1}$$

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$$\frac{\mathrm{d}v}{\mathrm{d}t}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu+F_y, \tag{2}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+F_z, \tag{3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{4}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=F_\theta. \tag{5}$$

其中(F_x, F_y, F_z)为涡动粘性项，F_θ为热量扩散项。

假设$u=\overline{U}+u'$，$v=v'$，$w=w'$，$p=\overline{P}+p'$，$\rho=\bar{\rho}+\rho'$和$\theta=\bar{\theta}+\theta'$代入到方程组(1)～(5)中，定义大气稳定度参数$N^2=\frac{g}{\bar\theta}\frac{\partial\bar\theta}{\partial z}$，$F^2=f\left(f-\frac{\partial\overline{U}}{\partial y}\right)$，$S_1^2=-f\frac{\partial\overline{U}}{\partial z}$，$S_2^2=\frac{g}{\bar\theta}\frac{\partial\bar\theta}{\partial y}$，采用Boussinesq近似，并且假定基本场满足地转平衡、静力平衡，物理量关于纬向x轴具有对称性($\partial/\partial x=0$)，则方程组(1)～(5)的线性化形式为

$$\frac{\partial}{\partial t}(fu')=F^2v'+S_1^2w'+fF'_x, \tag{6}$$

$$\frac{\partial v'}{\partial t}=-\frac{1}{\bar\rho}\frac{\partial p'}{\partial y}-fu'+F'_y, \tag{7}$$

$$\frac{\partial w'}{\partial t}=-\frac{1}{\bar\rho}\frac{\partial p'}{\partial z}+\frac{\theta'}{\bar\theta}g+F'_z, \tag{8}$$

$$\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0, \tag{9}$$

$$\frac{\partial}{\partial t}\left(\frac{\theta'}{\bar\theta}g\right)=-S_2^2v'-N^2w'+\frac{g}{\bar\theta}F'_\theta. \tag{10}$$

假定摩擦项$F'_x=\nu\nabla^2u'$，$F'_y=\nu\nabla^2v'$，$F'_z=\nu\nabla^2w'$，$F'_\theta=\kappa\nabla^2\theta'$（$\nu$为涡动粘性系数，$\kappa$为热量扩散系数），为了方便起见，此处进一步假设Prandtl数$Pr=\nu/\kappa=1$。由(9)式可见，引进扰动流函数ψ，使得$(v', w')=\left(-\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial y}\right)$，这样方程组(6)～(10)就简化成

$$\frac{\partial}{\partial t}(fu')=-F^2\frac{\partial\psi}{\partial z}+S_1^2\frac{\partial\psi}{\partial y}+\nu\nabla^2(fu'), \tag{11}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial z}\right)=\frac{1}{\bar\rho}\frac{\partial p'}{\partial y}+fu'+\nu\nabla^2\left(\frac{\partial\psi}{\partial z}\right), \tag{12}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial y}\right)=-\frac{1}{\bar\rho}\frac{\partial p'}{\partial z}+\frac{\theta'}{\bar\theta}g+\nu\nabla^2\left(\frac{\partial\psi}{\partial y}\right), \tag{13}$$

$$\frac{\partial}{\partial t}\left(\frac{\theta'}{\bar\theta}g\right)=S_2^2\frac{\partial\psi}{\partial z}-N^2\frac{\partial\psi}{\partial y}+\nu\frac{g}{\bar\theta}\nabla^2\theta'. \tag{14}$$

在方程组(11)～(14)中，消去变量u'、p'和θ'，就会得到关于流函数ψ的唯一变量方程

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)^2\nabla^2\psi=-N^2\frac{\partial^2\psi}{\partial y^2}+(S_1^2+S_2^2)\frac{\partial^2\psi}{\partial y\partial z}-F^2\frac{\partial^2\psi}{\partial z^2}$$
$$+\left(\frac{\partial S_1^2}{\partial z}-\frac{\partial N^2}{\partial y}\right)\frac{\partial\psi}{\partial y}+\left(\frac{\partial S_2^2}{\partial y}-\frac{\partial F^2}{\partial z}\right)\frac{\partial\psi}{\partial z}. \tag{15}$$

其中等式左边的 Laplace 算子展开以后得到

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)^2\nabla^2=\frac{\partial^4}{\partial t^2\partial y^2}+\frac{\partial^4}{\partial t^2\partial z^2}-2\nu\frac{\partial^5}{\partial t\partial y^4}-4\nu\frac{\partial^5}{\partial t\partial y^2\partial z^2}-2\nu\frac{\partial^5}{\partial t\partial z^4}$$
$$+\nu^2\frac{\partial^6}{\partial y^6}+3\nu^2\frac{\partial^6}{\partial y^4\partial z^2}+3\nu^2\frac{\partial^6}{\partial y^2\partial z^4}+\nu^2\frac{\partial^6}{\partial z^6}. \tag{16}$$

3　扰动波包方程的求解

假设基本场稳定度参数N^2、S_1^2、S_2^2、F^2仅是(y, z, t)的缓变函数，引进一组缓变变量$Y=\varepsilon y$，$Z=\varepsilon z$，$T=\varepsilon t$，流函数ψ具有如下波包形式:

$$\psi=A(Y,Z,T)e^{i\varepsilon^{-1}\theta(Y,Z,T)}, \tag{17}$$

其中$\theta=mY+nZ-\omega T$，ε为小参数。

将上式代入到(15)式中，并利用 WKBJ 方法，把振幅A按ε进行幂级数展开，即令

$$A=A_0+\varepsilon A_1+\varepsilon^2A_2+\cdots\cdots, \tag{18}$$

从而在方程(15)中，按ε幂次集项，精确到最低阶，得零级近似方程为

$$(m^2+n^2)[\omega+i\nu(m^2+n^2)]^2=N^2m^2-(S_1^2+S_2^2)mn+F^2n^2. \tag{19}$$

此式即为频散关系式，定义$l^2=m^2+n^2$，则(19)式可写成如下形式:

$$\omega=\pm\left[\frac{N^2m^2-(S_1^2+S_2^2)mn+F^2n^2}{l^2}\right]^{1/2}-i\nu l^2\triangleq\varphi(m,n)-i\nu l^2. \tag{20}$$

在波动稳定时，φ为实数，此时群速度为

$$C_{gy}=C_{gy}^r+iC_{gy}^i=\frac{\partial\varphi}{\partial m}-2i\nu m, \tag{21}$$

$$C_{gz}=C_{gz}^r+iC_{gz}^i=\frac{\partial\varphi}{\partial n}-2i\nu n, \tag{22}$$

这里$(C_{gy}^r$、$C_{gz}^r)$为群速度的实部，我们称之为“实群速度”；(C_{gy}^i, C_{gz}^i)为群速度的虚部，则称之为“虚群速度”，具体物理意义我们在下面给予论述。

在波动不稳定时，φ为纯虚数[可以证明，此时的条件为$(S_1^2+S_2^2)^2-4N^2F^2>0$]，不妨将$\varphi$写成$\varphi=\sigma_{\min}i$的形式，则

$$\omega=(\sigma_{\min}-\nu l^2)i, \tag{23}$$

此时群速度为

$$C_{gy}=iC_{gy}^i=i\left(\frac{\partial\sigma_{\min}}{\partial m}-2\nu m\right), \tag{24}$$

$$C_{gz}=\mathrm{i}C_{gz}^{\mathrm{i}}=\mathrm{i}\left(\frac{\partial\sigma_{\min}}{\partial n}-2vn\right).\tag{25}$$

注意此时$C_{gy}^{\mathrm{r}}=0$, $C_{gz}^{\mathrm{r}}=0$。

方程(15)精确到$O(\varepsilon^{r})$的近似方程为

$$\begin{aligned}&2l^2(\omega+\mathrm{i}vl^2)\frac{\partial A_0}{\partial T}+[-2m\omega^2+2mN^2-n(S_1^2+S_2^2)-8\mathrm{i}vml^2\omega+6v^2ml^4]\frac{\partial A_0}{\partial Y}\\&+[-2n\omega^2+2nF^2-m(S_1^2+S_2^2)-8\mathrm{i}vnl^2\omega+6v^2nl^4]\frac{\partial A_0}{\partial Z}\\&+\left\{2(\omega+\mathrm{i}vl^2)\frac{\partial l^2}{\partial T}+l^2\frac{\partial}{\partial T}(\omega+\mathrm{i}vl^2)-(\omega+\mathrm{i}vl^2)^2\left(\frac{\partial m}{\partial Y}+\frac{\partial n}{\partial Z}\right)+N^2\frac{\partial m}{\partial Y}\right.\\&+F^2\frac{\partial n}{\partial Z}-(S_1^2+S_2^2)\frac{\partial n}{\partial Y}-m\left(\frac{\partial S_1^2}{\partial Z}-\frac{\partial N^2}{\partial Y}\right)-n\left(\frac{\partial S_2^2}{\partial Y}-\frac{\partial F^2}{\partial Z}\right)\\&+\mathrm{i}vl^2\frac{\partial l^2}{\partial T}-2\mathrm{i}v(\omega+\mathrm{i}vl^2)\left[(5m^2+n^2)\frac{\partial m}{\partial Y}+(m^2+5n^2)\frac{\partial n}{\partial Z}+8mn\frac{\partial m}{\partial Z}\right]\\&\left.+4v^2l^2\left(m^2\frac{\partial m}{\partial Y}+n^2\frac{\partial n}{\partial Z}+2mn\frac{\partial m}{\partial Z}\right)\right\}A_0=0.\end{aligned}\tag{26}$$

注意到$C_{gy}=\partial\omega/\partial m$, $C_{gz}=\partial\omega/\partial n$, 上式中$\partial A_0/\partial Y$、$\partial A_0/\partial Z$的系数分别为$2l^2(\omega+\mathrm{i}vl^2)C_{gy}$、$2l^2(\omega+\mathrm{i}vl^2)C_{gz}$, 此外,

$$\omega+\mathrm{i}vl^2=\varphi,\tag{27}$$

$$(5m^2+n^2)\frac{\partial m}{\partial Y}+(m^2+5n^2)\frac{\partial n}{\partial Z}+8mn\frac{\partial m}{\partial Z}=l^2\left(\frac{\partial m}{\partial Y}+\frac{\partial n}{\partial Z}\right)+2m\frac{\partial l^2}{\partial Y}+2n\frac{\partial l^2}{\partial Z},\tag{28}$$

$$m^2\frac{\partial m}{\partial Y}+n^2\frac{\partial n}{\partial Z}+2mn\frac{\partial m}{\partial Z}=\frac{m}{2}\frac{\partial l^2}{\partial Y}+\frac{n}{2}\frac{\partial l^2}{\partial Z}.\tag{29}$$

于是, 方程(26)两边同乘以A_0以后就变换成

$$\begin{aligned}&\frac{\partial l^2A_0^2}{\partial T}+\nabla\cdot(l^2A_0^2\vec{C}_g)-A_0^2\nabla\cdot(l^2\vec{C}_g)+A_0^2\frac{\partial l^2}{\partial T}+\frac{A_0^2}{\varphi}\left\{l^2\frac{\partial\varphi}{\partial T}-\varphi^2\left(\frac{\partial m}{\partial Y}+\frac{\partial n}{\partial Z}\right)\right.\\&+N^2\frac{\partial m}{\partial Y}+F^2\frac{\partial n}{\partial Z}-(S_1^2+S_2^2)\frac{\partial n}{\partial Y}-m\left(\frac{\partial S^2}{\partial Z}-\frac{\partial N^2}{\partial Y}\right)-n\left(\frac{\partial S_2^2}{\partial Y}-\frac{\partial F^2}{\partial Z}\right)\\&+\mathrm{i}vl^2\frac{\partial l^2}{\partial T}-2\mathrm{i}v\varphi\left[l^2\left(\frac{\partial m}{\partial Y}+\frac{\partial n}{\partial Z}\right)+2m\frac{\partial l^2}{\partial Y}+2n\frac{\partial l^2}{\partial Z}\right]\\&\left.+2v^2l^2\left(m\frac{\partial l^2}{\partial Y}+n\frac{\partial l^2}{\partial Z}\right)\right\}=0.\end{aligned}\tag{30}$$

再对$\nabla\cdot(l^2\vec{C}_g)$和$l^2\partial\varphi/\partial T$进行运算, 代入上式并大量化简, 可以得到

$$\begin{aligned}&\frac{\partial l^2A_0^2}{\partial T}+\nabla\cdot(l^2A_0^2\vec{C}_g)+\frac{A_0^2}{\varphi}\left\{\frac{1}{2\varphi}\left[m^2\frac{\partial N^2}{\partial T}-mn\frac{\partial(S_1^2+S_2^2)}{\partial T}+n^2\frac{\partial F^2}{\partial T}\right]\right.\\&+\frac{m}{2}\left(\frac{\partial N^2}{\partial Y}-\frac{\partial S_1^2}{\partial Z}\right)+\frac{n}{2}\left(\frac{\partial F^2}{\partial Z}-\frac{\partial S_2^2}{\partial Y}\right)+\frac{\mathrm{i}v}{\varphi}\left(N^2m\frac{\partial l^2}{\partial Y}+F^2n\frac{\partial l^2}{\partial Z}\right)\end{aligned}$$

$$- \frac{\mathrm{i}\nu(S_1^2 + S_2^2)}{2\varphi}\left(n^2\frac{\partial l^2}{\partial Y} + m\frac{\partial l^2}{\partial Z}\right) + \mathrm{i}\nu l^2\frac{\partial l^2}{\partial T} - \mathrm{i}\nu\varphi\left(m\frac{\partial l^2}{\partial y} + n\frac{\partial l^2}{\partial Z}\right)$$
$$+ 2\nu^2 l^2\left(m\frac{\partial l^2}{\partial Y} + n\frac{\partial l^2}{\partial Z}\right)\Bigg\} = 0. \tag{31}$$

4　均匀介质中解的特征

为了简单起见，我们首先在均匀介质中讨论，并且假定波参数m、n、ω为常数，则(26)式退化为

$$\frac{\partial A_0}{\partial T} + \left(C_{gy}^{r} + \mathrm{i}C_{gy}^{i}\right)\frac{\partial A_0}{\partial Y} + \left(C_{gz}^{r} + \mathrm{i}C_{gz}^{i}\right)\frac{\partial A_0}{\partial Z} = 0. \tag{32}$$

令$A_0 = |A_0|e^{i\beta}$，$\frac{D_g}{DT} = \frac{\partial}{\partial T} + C_{gy}^{r}\frac{\partial}{\partial Y} + C_{gz}^{r}\frac{\partial}{\partial Z}$，代入上式可得到偏微分方程组，

$$\frac{D_g}{DT}\ln|A_0| = C_{gy}^{i}\frac{\partial\beta}{\partial Y} + C_{gz}^{i}\frac{\partial\beta}{\partial Z}. \tag{33}$$

$$\frac{D_g}{DT}\beta = -C_{gy}^{i}\frac{\partial\ln|A_0|}{\partial Y} - C_{gz}^{i}\frac{\partial\ln|A_0|}{\partial Z}. \tag{34}$$

标志$|C_{gy}|^2 = (C_{gy}^{r})^2 + (C_{gy}^{i})^2$，$|C_{gz}|^2 = (C_{gz}^{r})^2 + (C_{gz}^{i})^2$，将(33)和(34)式化为单一变量的微分方程

$$\left[\frac{\partial^2}{\partial T^2} + |C_{gy}|^2\frac{\partial^2}{\partial Y^2} + |C_{gz}|^2\frac{\partial^2}{\partial Z^2} + 2C_{gy}^{r}\frac{\partial^2}{\partial T\partial Y} + 2C_{gz}^{r}\frac{\partial^2}{\partial T\partial Z}\right.$$
$$\left. + 2(C_{gy}^{r}C_{gz}^{r} + C_{gy}^{i}C_{gz}^{i})\frac{\partial^2}{\partial Y\partial Z}\right]\ln|A_0| = 0. \tag{35}$$

仅考虑沿 y 方向的一维情况时，

$$\frac{\partial^2\ln|A_0|}{\partial T^2} + 2C_{gy}^{r}\frac{\partial^2\ln|A_0|}{\partial T\partial Y} + \left[(C_{gy}^{r})^2 + (C_{gy}^{i})^2\right]\frac{\partial^2\ln|A_0|}{\partial Y^2} = 0. \tag{36}$$

由于判别式$\Delta = -(C_{gy}^{i})^2 < 0$，故方程(36)对变量$(T, Y)$而言，是椭圆型偏微分方程。它的特征方程为$\mathrm{d}Y/\mathrm{d}T = C_{gy}^{r} + \mathrm{i}C_{gy}^{i}$，虚特征曲线为$Y + C_{gy}^{r}T - C_{gy}^{i}T = C$，作自变量变换$\xi = Y - C_{gy}^{r}T$，$\eta = -C_{gy}^{i}T$，则方程(36)化为标准型

$$\frac{\partial^2\ln|A_0|}{\partial\xi^2} + \frac{\partial^2\ln|A_0|}{\partial\eta^2} = 0. \tag{37}$$

令$\ln|A_0| = F_1(\xi)F_2(\eta)$，就有$F_1''/F_1 = -F_2''/F_2 \triangleq -\lambda^2$（$\lambda$为大于零的常数），可解得

$$\ln|A_0| = [C_1\mathrm{ch}(-\lambda C_{gy}^{i}T) + C_2\mathrm{sh}(-\lambda C_{gy}^{i}T)]\cdot[C_3\cos\lambda(Y - C_{gy}^{r}T)$$
$$+ C_4\sin\lambda(Y - C_{gy}^{r}T)]. \tag{38}$$

将(18)、(20)两式代入(17)式，在零级近似下，

$$\psi = A_0 \exp\{i(mY + nZ - \varphi T)/\varepsilon - \nu l^2 T/\varepsilon\}$$
$$= |A_0| \exp(-\nu l^2 t) \exp\{i(my + nz - \varphi t + \beta)\}. \tag{39}$$

综合(38)和(39)两式可见，由于在动力学模型中引入粘性耗散及扩散，导致了"虚群速度(C^{i}_{gy}、C^{i}_{gz})"的产生。粘性的作用不仅使得扰动随时间呈指数衰减，它还使得波包迹的振幅大小随缓变时间变量 T 的变化（与虚群速度有关）。波包迹的传播速度为实群速度(C^{r}_{gy})。在波动不稳定时，φ 为纯虚数，此时$C^{r}_{gy}=0$，波包迹不传播、呈现驻波形式，但是C^{i}_{gy} 不为零，波包迹的振幅可以随缓变时间变量 T 增长，这称之为波包的不稳定现象。

对于(35)式考虑(Y、Z)二维情况时，作变量替换，$\xi = Y - C^{r}_{gy}T$、$\eta = Z - C^{r}_{gz}T$，$\tau = T$，则方程(35)变换为

$$\left[\frac{\partial^2}{\partial \tau^2} + (C^{i}_{gy})^2 \frac{\partial^2}{\partial \xi^2} + 2C^{i}_{gy}C^{i}_{gz}\frac{\partial^2}{\partial \xi \partial \eta} + (C^{i}_{gz})^2 \frac{\partial^2}{\partial \eta^2}\right] \ln|A_0| = 0. \tag{40}$$

令 $\ln|A_0| = F_1(\tau)F_2(\xi,\ \eta)$，进行分离变量，并标志算子 $L = (C^{i}_{gy})^2 \frac{\partial^2}{\partial \xi^2} + 2C^{i}_{gy}C^{i}_{gz}\frac{\partial^2}{\partial \xi \partial \eta} + (C^{i}_{gz})^2 \frac{\partial^2}{\partial \eta^2}$，就有

$$\frac{F''_1}{F_1} = -\frac{LF_2}{F_2} = \lambda^2. \tag{41}$$

此时可解得，

$$F_1 = C_1 \mathrm{ch}(\lambda\tau) + C_2 \mathrm{sh}(\lambda\tau), \tag{42}$$

$$(C^{i}_{gy})^2 \frac{\partial^2 F_2}{\partial \xi^2} + 2C^{i}_{gy}C^{i}_{gz}\frac{\partial^2 F_2}{\partial \xi \partial \eta} + (C^{i}_{gz})^2 \frac{\partial^2 F_2}{\partial \eta^2} + \lambda^2 F_2 = 0. \tag{43}$$

对于偏微分方程(43)式，判别式 $\Delta = 0$，故它的类型是抛物型，其特征方程为$\mathrm{d}\eta/\mathrm{d}\xi = C^{i}_{gz}/C^{i}_{gy}$，特征曲线为$C^{i}_{gz}\xi - C^{i}_{gy}\eta = C$，作坐标变换$\zeta_1 = C^{i}_{gz}\xi - C^{i}_{gy}\eta$，$\zeta_2 = a\xi + b\eta$（注意常数 a 和 b 满足$aC^{i}_{gy} + bC^{i}_{gz} \neq 0$），则方程(43)式变换为

$$[aC^{i}_{gy} + bC^{i}_{gz}]^2 \frac{\partial^2 F_2}{\partial \zeta_2^2} + \lambda^2 F_2 = 0. \tag{44}$$

令$F_2 = F_3(\zeta_1)F_4(\zeta_2)$，得到关于$F_4$的常微分方程[此处标志$\mu = \lambda/(aC^{i}_{gy} + bC^{i}_{gz})$]，

$$\frac{\mathrm{d}^2 F_4}{\mathrm{d}\zeta^2} + \mu^2 F_4 = 0. \tag{45}$$

方程(45)的解为

$$F_4 = C_3 \cos\mu(a\xi + b\eta) + C_4 \sin\mu(a\xi + b\eta). \tag{46}$$

综合(42)与(46)两式，

$$\ln|A_0| = [C_1 \mathrm{ch}\mu(aC^{i}_{gy}T + bC^{i}_{gz}T) + C_2 \mathrm{sh}\mu(aC^{i}_{gy}T + bC^{i}_{gz}T)]$$
$$\times F_3[C^{i}_{gz}(Y - C^{r}_{gy}T) - C^{i}_{gy}(Z - C^{r}_{gz}T)] \cdot \{C_3 \cos\mu[a(Y - C^{r}_{gy}T)$$

$$+ b(Z - C_{gz}^{r} T)] + C_4 \sin\mu[a(Y - C_{gy}^{r} T) + b(Z - C_{gz}^{r} T)]\}. \tag{47}$$

仔细分析一下其物理意义，可以得到类似于一维情况的结论，只不过F_3函数的产生与方程(35)中的混合偏导数$\partial^2 / \partial Y \partial Z$有关。

事实上，以上的分析结论可以通过简单的谐波叠加来加以验证。考虑两列振幅相同，圆频率，波数略有差别的简谐波。设其振幅为 A，圆频率分别为$\omega - \delta\omega$、$\omega + \delta\omega$，波数分别为$m - \delta m$，$n - \delta n$；$m + \delta m$，$n + \delta n$，这两列等幅简谐波相叠加，其合成波为

$$f(y,z,t) = A\{e^{i[(m-\delta m)y + (n-\delta n)z - (\omega-\delta\omega)t]} + e^{i[(m+\delta m)y + (n+\delta n)z - (\omega+\delta\omega)t]}\}. \tag{48}$$

注意到$\omega = \omega_r + i\omega_i$、$\delta\omega_r = C_{gy}^{r}\delta m + C_{gz}^{r}\delta n$，$\delta\omega_i = C_{gy}^{i}\delta m + C_{gz}^{i}\delta n$，上式经过简单运算后，可改写为

$$f(y,z,t) = 2Ae^{\omega_i t}e^{i(my + nz + \omega_r t)}\{\mathrm{ch}(C_{gy}^{i} t\delta m + C_{gz}^{i} t\delta n)\cos[(y - C_{gy}^{r} t)\delta m + (z - C_{gz}^{r} t)\delta n]$$
$$+ i\,\mathrm{sh}(C_{gy}^{i} t\delta m + C_{gz}^{i} t\delta n)\sin[(y - C_{gy}^{r} t)\delta m + (z - C_{gz}^{r} t)\delta n]. \tag{49}$$

可见，波包迹是以实群速度(C_{gy}^{r}, C_{gz}^{r})传播，虚群速度(C_{gy}^{i}, C_{gz}^{i})对波包迹振幅的大小有影响（呈现双曲函数变化）。

5 非均匀介质中波包的发展

下面对方程(31)分两种情况考虑:

(1) 在波动稳定的情况下，φ 为实数，$C_{gy} = C_{gy}^{r} + iC_{gy}^{i} = \dfrac{\partial\varphi}{\partial m} - 2i\nu m$，$C_{gz} = C_{gz}^{r} + iC_{gz}^{i} = \dfrac{\partial\varphi}{\partial n} - 2i\nu n$，并令$A_0 = |A_0|e^{i\beta}$，代入到(31)式中，分离实部、虚部后得到

$$\frac{D_g(l^2|A_0|^2)}{DT} + l^2|A_0|^2\nabla\cdot\vec{C}_g^{r} = -\frac{|A_0|^2}{2\varphi^2}\left[m^2\frac{\partial N^2}{\partial T} - mn\frac{\partial(S_1^2 + S_2^2)}{\partial T} + n^2\frac{\partial F^2}{\partial T}\right]$$
$$-\frac{m|A_0|^2}{2\varphi}\left(\frac{\partial N^2}{\partial Y} - \frac{\partial S_1^2}{\partial Z}\right) - \frac{n|A_0|^2}{2\varphi}\left(\frac{\partial F^2}{\partial Z} - \frac{\partial S_2^2}{\partial Y}\right)$$
$$-\frac{2\nu^2 l^2|A_0|^2}{\varphi}\left(m\frac{\partial l^2}{\partial Y} + n\frac{\partial l^2}{\partial Z}\right) - 4\nu l^2|A_0|^2\left(m\frac{\partial\beta}{\partial Y} + n\frac{\partial\beta}{\partial Z}\right). \tag{50}$$

$$\frac{D_g\beta}{DT} = -\frac{\nu}{2l^2\varphi^2}\left(N^2 m\frac{\partial l^2}{\partial Y} + F^2 n\frac{\partial l^2}{\partial Z}\right) + \frac{\nu(S_1^2 + S_2^2)}{4l^2\varphi^2}\left(n\frac{\partial l^2}{\partial Y} + m\frac{\partial l^2}{\partial Z}\right) - \frac{\nu}{2\varphi}\frac{\partial l^2}{\partial T}$$
$$+ \frac{\nu}{2l^2}\left(m\frac{\partial l^2}{\partial Y} + n\frac{\partial l^2}{\partial Z}\right) + \frac{\nu}{l^2|A_0|^2}\left[\frac{\partial}{\partial Y}(ml^2|A_0|^2) + \frac{\partial}{\partial Z}(nl^2|A_0|^2)\right], \tag{51}$$

其中$\dfrac{D_g}{DT} = \dfrac{\partial}{\partial T} + C_{gy}^{r}\dfrac{\partial}{\partial Y} + C_{gz}^{r}\dfrac{\partial}{\partial Z}$，$\vec{C}_g^{r} = C_{gy}^{r}\vec{j} + C_{gz}^{r}\vec{k}$。

定义一变量$E = \dfrac{1}{4}l^2|A_0|^2$（注意，在无粘情况下，$E$表示波能密度。在存在粘性时，由于振幅随时间呈指数衰减，波能密度则不能用E表示，不过波包的发展与E有

关)，则(50)式变换为

$$\frac{D_g E}{DT}+E\nabla\cdot\vec{C}_g=-\frac{|A_0|^2}{8\varphi^2}\left[m^2\frac{\partial N^2}{\partial T}-mn\frac{\partial(S_1^2+S_2^2)}{\partial T}+n^2\frac{\partial F^2}{\partial T}\right]$$
$$-\frac{m|A_0|^2}{8\varphi}\left(\frac{\partial N^2}{\partial Y}-\frac{\partial S_1^2}{\partial Z}\right)-\frac{n|A_0|^2}{8\varphi}\cdot\left(\frac{\partial F^2}{\partial Z}-\frac{\partial S_2^2}{\partial Y}\right)$$
$$-\frac{v^2l^2|A_0|^2}{2\varphi}\left(m\frac{\partial l^2}{\partial Y}+n\frac{\partial l^2}{\partial Z}\right)-vl^2|A_0|^2\left(m\frac{\partial\beta}{\partial y}+n\frac{\partial\beta}{\partial Z}\right). \tag{52}$$

记$\omega=\Omega(m,n,Y,Z,T)=\varphi(m,n,Y,Z,T)-\mathrm{i}vl^2$，代入到如下两式中

$$\frac{\partial m}{\partial T}+C_{gy}\frac{\partial m}{\partial Y}+C_{gz}\frac{\partial m}{\partial Z}=-\frac{\partial\Omega}{\partial Y}, \tag{53}$$

$$\frac{\partial n}{\partial T}+C_{gy}\frac{\partial n}{\partial Y}+C_{gz}\frac{\partial n}{\partial Z}=-\frac{\partial\Omega}{\partial Z}. \tag{54}$$

再分离实部、虚部，就有

$$\frac{D_g m}{DT}=\frac{\partial m}{\partial T}+C_{gy}^{\mathrm{r}}\frac{\partial m}{\partial Y}+C_{gz}^{\mathrm{r}}\frac{\partial m}{\partial Z}=-\frac{\partial\varphi}{\partial Y}, \tag{55}$$

$$\frac{D_g n}{DT}=\frac{\partial n}{\partial T}+C_{gy}^{\mathrm{r}}\frac{\partial n}{\partial Y}+C_{gz}^{\mathrm{r}}\frac{\partial n}{\partial Z}=-\frac{\partial\varphi}{\partial Z}, \tag{56}$$

$$\frac{\partial l^2}{\partial Y}=0, \qquad \frac{\partial l^2}{\partial Z}=0. \tag{57}$$

将(57)式代入到(52)、(51)式之中，则该两式简化为

$$\frac{D_g E}{DT}+E\nabla\cdot\vec{C}_g=-\frac{|A_0|^2}{8\varphi^2}\left[m^2\frac{\partial N^2}{\partial T}-mn\frac{\partial(S_1^2+S_2^2)}{\partial T}+n^2\frac{\partial F^2}{\partial T}\right]$$
$$-\frac{m|A_0|^2}{8\varphi}\left(\frac{\partial N^2}{\partial Y}-\frac{\partial S_1^2}{\partial Z}\right)-\frac{n|A_0|^2}{8\varphi}\left(\frac{\partial F^2}{\partial Z}-\frac{\partial S_2^2}{\partial y}\right)$$
$$-vl^2|A_0|^2\left(m\frac{\partial\beta}{\partial Y}+n\frac{\partial\beta}{\partial Z}\right), \tag{58}$$

$$\frac{D_g\beta}{DT}=-\frac{v}{2\varphi}\frac{\partial l^2}{\partial T}+\frac{v}{|A_0|^2}\left[\frac{\partial}{\partial Y}(m|A_0|^2)+\frac{\partial}{\partial Z}(n|A_0|^2)\right]. \tag{59}$$

分析(58)式，不失一般性取$\varphi>0$进行讨论，由于扰动的等位相线随高度向北（冷区）倾斜，并且假设等位相线向南传播（$m<0$，$n>0$，这与我国大部分地区带状扰动向南传播的情形是一致的），在背景场处于热成风平衡的情况下，波包的发展与基本场的非定常性[5]以及粘性作用有关。若背景场处于非热成风平衡时，则扰动波包发展所需要的条件是：（1）静力稳定度参数N^2随时间减小；（2）大气的斜压稳定度参数(S_1^2, S_2^2)随时间减小，亦即大气的斜压性$(\partial\overline{U}/\partial Z, -\partial\overline{\theta}/\partial Y)$随着时间越来越强；（3）惯性稳定度$F^2$随时间越来越小；（4）$\partial N^2/\partial Y>0$，亦即波动向南传播过程中，移向层结稳定度的低值区，扰动就会发展；（5）$\partial F^2/\partial Z<0$，亦即波动向上传播的过程中，移向惯性稳定度的低值区，扰动会发展；（6）$\partial S_1^2/\partial Z<0$，它反映了纬向基本气流随高度的二阶变化，当低层风速的垂直切变较小，而高层的风速垂直切变较大时，甚

至于出现低空急流，此时扰动波包将发展起来；（7）$\partial S_2^2 / \partial Y > 0$，它反映了基本场位温在南北方向上的二阶变化，当位温$\bar{\theta}$在某一纬度Y_0上出现最小值时，这样的基本场位温分布有利于扰动波包的发展；（8）粘性耗散及扩散对波包发展的作用是通过波幅包络线位相的空间分布来体现的，在 $v \neq 0$ 的情况下，当波包迹的传播方向与波动等位相线的传播方向相同时，亦即波包迹偏向于向南向上传播($\partial\beta / \partial Y < 0$，$\partial\beta / \partial Z > 0$)时，此时粘性的作用明显地使波包衰减下去，其具体动力学过程必须通过联立求解偏微分方程组(58)和(59)来获得，求解过程十分复杂，此处暂不予讨论。

（2）在波动不稳定的情况下，φ 为纯虚数，$\varphi = \sigma_{\min}\mathrm{i}$，$C_{gy} = (\partial\sigma_{\min} / \partial m - (\partial\sigma_{\min} / \partial m - 2vm)\mathrm{i}$，$C_{gz} = (\partial\sigma_{\min} / \partial n - 2vn)\mathrm{i}$，并令$A_0 = |A_0|\mathrm{e}^{\mathrm{i}\beta}$，代入到(31)式中，分离实部，虚部后得到

$$\frac{\partial E}{\partial T} = \frac{|A_0|^2}{8\sigma_{\min}^2}\left[m^2\frac{\partial N^2}{\partial T} - mn\frac{\partial(S_1^2 + S_2^2)}{\partial T} + n^2\frac{\partial F^2}{\partial T}\right] - \frac{vl^2|A_0|^2}{4\sigma_{\min}}\frac{\partial l^2}{\partial T}$$
$$+ \frac{l^2|A_0|^2}{2}\left[\left(\frac{\partial\sigma_{\min}}{\partial m} - 2vm\right)\frac{\partial\beta}{\partial Y} + \left(\frac{\partial\sigma_{\min}}{\partial n} - 2vn\right)\frac{\partial\beta}{\partial Z}\right], \tag{60}$$

$$\frac{\partial\beta}{\partial T} = \frac{m}{4\sigma_{\min}l^2}\left(\frac{\partial N^2}{\partial Y} - \frac{\partial S_1^2}{\partial Z}\right) + \frac{n}{4\sigma_{\min}l^2}\left(\frac{\partial F^2}{\partial Z} - \frac{\partial S_2^2}{\partial Y}\right)$$
$$+ \frac{v}{2\sigma_{\min}l^2}\left(N^2 m\frac{\partial l^2}{\partial Y} + F^2 n\frac{\partial l^2}{\partial Z}\right) - \frac{v(S_1^2 + S_2^2)}{4\sigma_{\min}^2 l^2}\left(n\frac{\partial l^2}{\partial Y} + m\frac{\partial l^2}{\partial Z}\right)$$
$$+ v\left(\frac{1}{2l^2} + \frac{v}{\sigma_{\min}}\right)\left(m\frac{\partial l^2}{\partial Y} + n\frac{\partial l^2}{\partial Z}\right) - \frac{1}{2l^2|A_0|^2}\left\{\frac{\partial}{\partial Y}\left[l^2|A_0|^2\left(\frac{\partial\sigma_{\min}}{\partial m}\right.\right.\right.$$
$$\left.\left.\left.- 2vm\right)\right] + \frac{\partial}{\partial Z}\left[l^2|A_0|^2\left(\frac{\partial\sigma_{\min}}{\partial n} - 2vn\right)\right]\right\}. \tag{61}$$

同样记$\omega = \Omega(m, n, Y, Z, T) = (\sigma_{\min} - vl^2)\mathrm{i}$，代入到(53)和(54)式中，并且分离实部，虚部就有

$$\left(\frac{\partial\sigma_{\min}}{\partial m} - 2vm\right)\frac{\partial m}{\partial Y} + \left(\frac{\partial\sigma_{\min}}{\partial n} - 2vn\right)\frac{\partial m}{\partial Z} = -\frac{\partial\sigma_{\min}}{\partial Y}, \tag{62}$$

$$\left(\frac{\partial\sigma_{\min}}{\partial m} - 2vm\right)\frac{\partial n}{\partial Y} + \left(\frac{\partial\sigma_{\min}}{\partial n} - 2vn\right)\frac{\partial n}{\partial Z} = -\frac{\partial\sigma_{\min}}{\partial Z}, \tag{63}$$

$$\frac{\partial m}{\partial T} = 0, \qquad \frac{\partial n}{\partial T} = 0. \tag{64}$$

将(64)式代入到(60)式中，则(60)式中$\partial l^2 / \partial T$这一项消失。从(60)式可以看出，$\partial N^2 / \partial T < 0$，$\partial F^2 / \partial T < 0$，即层结稳定度和惯性稳定度随时间减小时，扰动波包并非随时间发展，而是随时间衰减下去。也就是说，此时静力稳定度和惯性稳定度的减小反而抑制了波包的发展，这与前面稳定的波包($\varphi^2 > 0$)得出的结论是大不相同的！此外，从(61)式可以看出，各种稳定度参数的空间分布不均匀只对波幅包络线位相的变化起作用，而对波包的发展并没有影响。

6 结论

本文首先将粘性($Pr=1$)引进中尺度对称扰动波包的动力学方程组中，粘性耗散及扩散导致了"虚群速度(C^{i}_{gy}，C^{i}_{gz})"的产生，它不仅使得扰动随时间呈指数衰减，还使得波包迹的振幅大小随着缓变时间变量 T 变化（与虚群速度有关）。波包迹的传播速度则为实群速度(C^{r}_{gy}，C^{r}_{gz})。在一定条件下，波包迹不传播($C^{r}_{gy}=C^{r}_{gz}=0$)，呈现驻波形式，但是波包迹的振幅可以随缓变时间变量 T 增长，这称之为波包的不稳定现象。此外，各种稳定度参数的时空变化对扰动波包的发展具有重要作用。对于稳定和不稳定的波包而言，静力稳定度和惯性稳定度随着时间的减小对波包的发展所起作用是完全不相同的，前者使得波包发展加强，后者则使得波包衰减下去。

为了将粘性耗散与热量扩散的作用区分开，Prandtl 数 $Pr\neq 1$ 情况下粘性波包的发展理论研究是我们今后要进行的课题。

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Interactions between Symmetric Disturbance and Zonally Basic Flow Part II: The Development and Dispersion of the Viscosity Wave Packet

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Abstract The present paper has comprehensively studied the problems of the interactions between symmetric disturbance and zonally basic flow. Here is its Part II. Mainly dealing with the development and dispersion of the viscosity wave packet. It is found by introducing viscosity ($Pr=1$) into the dynamic equations of symmetric disturbance wave packet that the dissipation and diffusion of viscosity lead to the producing of " imaginary dispersion velocity (C^{i}_{gy}, C^{i}_{gy})" which has a great effect on development of wave packet. The spacial and temporal changes of each kind of stability parameter play an importand role in development of the disturbance wave packet. The phenomenon of instability of wave packet may be produced on some conditions.

Key words wave-flow interaction viscosity wave packet